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# Extended Ladder Representations

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*Abstract*

We analyse the ladder representations from an algebraic point of view. We find that these representations are closely connected to the representation of the left multiplication of the algebra of all polynomials $U/J$ of the creation and annihilation operators. It turns out that the ladder representations appear as a projection of the left multiplications on a subspace of $U/J$. The algebraic approach allows us to extend the class of the ladder representations.

## 1. *Introduction*

The ladder representations of a given Lie algebra $A$ are usually obtained through realisation of the generators of the algebra in terms of second-order polynomials of creation and annihilation operators† $b_i, a_j (i,j = 1,\dots,p)$, with subsequent realisation of this operators in the space $V$ of all polynomials of the creation operators (Todorov, 1966). The representations of $a_i, b_j$ used in this case is the same as in quantum field theory:

$$\begin{aligned} a_i \circ b_1^{n_1} \cdots b_i^{n_i} \cdots b_p^{n_p} |0\rangle &= n_i b_1^{n_1} \cdots b_i^{n_i - 1} \cdots b_p^{n_p} \\ b_i \circ b_1^{n_1} \cdots b_i^{n_i} \cdots b_p^{n_p} |0\rangle &= b_1^{n_1} \cdots b_i^{n_i+1} \cdots b_p^{n_p} \end{aligned} \tag{1}$$

where $|0\rangle$ is the vacuum state annihilated by $a_i$, i.e. $a_i|0\rangle = 0$. We shall refer to the representation (1) as the ladder representation of the creation and annihilation operators.

In the present paper we analyse the representations of the type (1) from a purely algebraic point of view. We find that the ladder representations appear as a projection of the left multiplications of the algebra $U/J$ of all polynomials of the canonical variables onto the subspace $V \subset U/J$ spanned on all monomials of the creation operators. The algebraic approach allows us to enlarge the representation space $V$ to the subspace $\tilde{V}$ of the quotient division ring of the Heisenberg algebra generated by $b_1, \dots, b_p$. In this way we obtain also some new ladder-type representations. We do not treat any topological properties of the representations, and only in an example do we show how one can introduce a metric in $\tilde{V}$ in order to obtain some of the hermitian representations of $SU(1,1)$. The theorem stated below

† These operators are also called Heisenberg algebra generators or canonical variables.

shows why the representations of the Heisenberg algebra are important and what their connection is with the representations of the Lie algebras.

*Theorem* (Doebner & Palev, 1970a)

The elements of an arbitrary finite Lie algebra $A$ may be realised as rational functions of creation and annihilation operators. In particular there always exists a realisation of $A$ in terms of polynomials of arbitrary order in $a_i$ and $b_j$ $(i,j = 1,\ldots,p)$.

Therefore, the bigger the class of representations of the canonical variables, the larger the class of representations of $A$ can be constructed. For this purpose representations in which $a_i$ is not the hermitian conjugate of $b_i$ are also useful.

In Section 2 we introduce the concept of a free algebra $U$ of arbitrary entities $a_1,\ldots,a_p$ over the field $F$ and define $G$-representations of $U$. In Section 3 we use these representations in order to construct the ladder-type representations of the creation and annihilation operators.

## 2. *Definitions and Preliminary Results*

Let $A = (a_1,\ldots,a_p)$ be given entities and $F$ be an arbitrary field. Denote by $U(A)$ the free algebra generated by $a_1,\ldots,a_p$ over $F$, i.e. the set of all polynomials of the entities $a_1,\ldots,a_p$ over $F$. Let $G = (g_i|i \in I)$ be a given linearly independent set of elements from $U(A)$ labelled by the index set $I$. By $J(G)$ we denote the ideal in $U(A)$ generated from $G$, and $U(A)/J(G)$ is the factor algebra of $U(A)$ modulo $J(G)$. We call the elements $g_i$, $i \in I$ structure relations of $U(A)/J(G)$.

Let $\theta$ be a mapping of the generators $a_i$ of $U(A)$ into a set $\tilde{a}_i$ of linear transformations in some linear space $L$ over $F$, i.e.

$$\theta : a_i \to \tilde{a}_i = \theta a_i \qquad (i = 1,\ldots,p) \tag{2}$$

For an arbitrary $h(a_1,\ldots,a_p) \in U(A)$ the mapping

$$h(a_1,\ldots,a_p) \to h(\theta a_1,\ldots,\theta a_n) \tag{3}$$

induces a homomorphism of $U(A)$ into the algebra $\varDelta$ of all linear transformations of $L$.

*Definition*

Let $\theta$ be the homomorphism of $U(A)$ into $\varDelta$ defined above and let $G = (g_i|i \in I)$ be a given set of elements from $U(A)$. We say that $\theta$ is a $G$-representation (or representation of the elements from $G$) of the free algebra $U(A)$ if $G$ is contained in the kernel $K$ of the homomorphism, i.e. if

$$\theta g_i \equiv g_i(\theta a_1,\ldots,\theta a_p) = 0 \qquad (i \in I) \tag{4}$$

*Lemma*

Let $\theta$ be a $G$-representation of $U(A)$ in $L$. Then $\theta$ defines a representation of the factor algebra $U(A)/J(G)$ in $L$.

*Proof:* We need only to show that $\theta$ considered as a mapping of $U(A)/J(K)$ into $\Delta$ is defined in the right way, that is if $a,b \in U(A)$ and $a = b(\text{mod } J)$ then $\theta a = \theta b$. Since the kernel of $\theta$ is an ideal, and $G \subset K$, we have $J(G) \subset K$. Suppose $a = b(\text{mod } J)$. Then $a - b = j \in J$.

$$\begin{aligned} \theta(a-b) = \theta(j) = 0 \\ \theta a = \theta b. \end{aligned} \tag{5}$$

The algebras $U(A)$ and $U(A)/J(G)$ are associative algebras. With respect to the composition $[a,b] = ab - ba$ they are Lie algebras, and $\theta$ is a Lie-homomorphism of $U(A)$ and $U(A)/J(G)$ into $\Delta$.

*Corollary*

Let $\varphi$ be a Lie-homomorphism of the Lie algebra $B$ in $U(A)/J(G)$. To every $G$-representation of $U(A)$ there corresponds a (Lie) representation $\theta\varphi$ of $B$ in $L$.

## 3. *Some Representations of the Structure Relations*

In the present section we discuss some representations of arbitrary structure relations. We apply the results to the Heisenberg structure relations and show how they can be used in order to extend the class of the ladder representations (1).

Let $G = (g_1, \ldots, g_r)$ be the structure relations of the algebra $U(A)/J(G)$. One natural $G$-representation of $U(A)$ can be constructed in the following way. Consider as representation space $L$ the algebra $U(A)/J(G)$. With every $a_i$ we associate a linear transformation $\tilde{a}_i = \theta a_i$ of $U(A)/J(G)$ as follows

$$\tilde{a}_i . l = a_i l \tag{6}$$

where $l \in U(A)/J(G)$ and the multiplication between $a_i$ and $l$ is the one defined in $U(A)$. The mapping $\theta : a_i \to \tilde{a}_i$ defines a representation of $U(A)$. This is the so-called representation by left multiplications. For every $h \in U(A)$ $g_i(a_1, \ldots, a_p)h \in J(G)$, since $g_i \in J(G)$. Therefore, $g_i(a_1, \ldots, a_p)h = 0(\text{mod } J)$, and we have

$$g_i(a_1, \ldots, a_p) . U(A)/J(G) = 0 \tag{7}$$

i.e., $\theta$ is $G$-representation of $U(A)$ and hence it defines a representation of $U(A)/J(G)$.

The representation of $U(A)/J(G)$ by left multiplications will help us to construct a class of new representations which in the case of the Heisenberg algebra is larger than the ladder representations (1). Consider a pair $(V, \varphi)$, where $V$ is a subspace of $U(A)/J(G)$ and $\varphi$ is a linear transformation defined in the image space $\theta V$ with the following properties

$$\begin{aligned} \varphi(\theta V) \subset V \\ g_i(\varphi\theta a_1, \ldots, \varphi\theta a_p) = 0 \end{aligned} \tag{8}$$

The mapping $\psi: a_i \to \tilde{a}_i = \varphi\theta . a_i$ associate with every generator $a_i$ a linear transformation $\tilde{a}_i$ in $V$. By construction $\psi$ is a $G$-representation. To every pair $(V,\varphi)$ there corresponds a $G$-representation of $U(A)$ in $V$ induced by the representation of the left multiplications.

We now proceed to study in more detail the structure relations corresponding to the Heisenberg algebra. Denote by $H$ the set of $2p$ generators $a_1,\dots,a_p$, $b_1,\dots,b_p$ of the free algebra $U(H)$ over the field $F$ and let $J(G)$ be an ideal in $U(H)$ generated by the following set $G$ of elements of $U(H)$

$$\begin{aligned} &a_i b_j - b_j a_i - 1\delta_{ij} \\ &a_i a_j - a_j a_i \\ &b_i b_j - b_j b_i \end{aligned} \tag{9}$$

where $i,j = 1,\dots,p$ and 1 is the unity of $U(H)$. In the factor algebra $U(H)/J(G)$ we have that

$$\begin{aligned} &[a_i, b_j] = 1\delta_{ij} \\ &[a_i, a_j] = [b_i, b_j] = 0 \end{aligned} \tag{10}$$

Thus the structure relations (9) lead to the commutation relations (10) for the Bose creation and annihilation operators. The subspace $H \subset U(H)/J(G)$ spanned on the operators $a_i$, $b_j$ and 1 is a Lie algebra with respect to the commutation relations (10), which is usually called a Heisenberg algebra. The associative algebra $U(H)/J(G)$ consists of all polynomials of the creation and annihilation operators. It differs from the universal enveloping algebra of $H$, since the latter is given by all polynomials of $a_i$, $b_j$ and 1 (there, $1^n \neq 1$). In a previous paper (Doebner & Palev, 1970b) we have shown that $U(H)/J(G)$ is a homomorphic image of the universal enveloping algebra. There we call it a quasi-enveloping algebra of $H$. The basis in $U(H)/J(G)$ is given by all ordered monomials

$$\prod_{i=1}^{p} b_i^{n_i} \prod_{j=1}^{p} a_j^{m_j} \tag{11}$$

Consider the quasi-enveloping algebra as a linear space over $F$. We obtain the representation $\theta$ of the structure relations (9) as left multiplications in $U(H)/J(G)$ if we define the operators $\tilde{a}_i = \theta a_i$ and $\tilde{b}_i = \theta b_i$ as follows:

$$\begin{aligned} \tilde{a}_i \circ b_1^{n_1} \cdots b_i^{n_i} \cdots b_p^{n_p} a_1^{m_1} \cdots a_p^{m_p} &= n_i b_1^{n_1} \cdots b_i^{n_i} \cdots b_p^{n_p} a_1^{m_1} \cdots a_p^{m_p} + \\ &+ b_1^{n_1} \cdots b_i^{n_i} \cdots b_p^{n_p} a_i^{m_i} \cdots a_i^{m_i+1} \cdots a_p^{m_p} \\ \tilde{b}_i \circ b_1^{n_1} \cdots b_i^{n_i} \cdots b_p^{n_p} a_1^{n_1} \cdots a_i^{m_i} \cdots a_p^{m_p} &= b_1^{n_1} \cdots b_i^{n_i+1} \cdots b_p^{n_p} a_1^{m_1} \cdots a_p^{m_p} \end{aligned} \tag{12}$$

Define now a pair $(V,\varphi)$ in the following way. Let $V$ be the subspace of $U(H)/J(G)$ spanned on all monomials of the creation operators $b_1,\dots,b_p$. Then $\theta V$ is the space of all linear combinations of the elements $b_1^{n_1}\dots b_p^{n_p}$ and $b_1^{n_1}\dots b_p^{n_p} a_i$, $i = 1,\dots,p$. Let $B_i$ be an arbitrary polynomial of $b_i$, $i = 1,\dots,p$. Define an operator $\varphi(B_1,\dots,B_p)$ in the subspace $\theta V$ to be

$$\begin{aligned} \varphi(B_1,\dots,B_p) \circ b_1^{n_1} \cdots b_p^{n_p} &= b_1^{n_1} \cdots b_p^{n_p} \\ \varphi(B_1,\dots,B_p) \circ b_1^{n_1} \cdots b_p^{n_p} a_i &= b_1^{n_1} \cdots b_p^{n_p} B_i \end{aligned} \tag{13}$$

The operator $\varphi(B_1,\ldots,B_p)$ satisfies the conditions (8), and therefore the mapping $\psi: a_i \to \bar{a}_i = \varphi\theta a_i$ and $b_j \to \bar{b}_j = \varphi\theta b_j$ defines a representation of the structure relations (9). Indeed, from the first of the relations (13) we have that $\varphi(B_1,\ldots,B_p).\ \theta V = V$. The proof of equation (8) is straightforward. If $i = j$

$$\begin{aligned}(\bar{a}_i\bar{b}_j - \bar{b}_j\bar{a}_i) \circ b_1^{n_1}\cdots b_i^{n_i}\cdots b_j^{n_j}\cdots b_p^{n_p} &= \bar{a}_i \circ b_1^{n_1}\cdots b_i^{n_i}\cdots b_j^{n_j+1}\cdots b_p^{n_p} - \\ &- \bar{b}_j \circ [n_i b_1^{n_1}\cdots b_i^{n_i-1}\cdots b_j^{n_j}\cdots b_p^{n_p} + b_1^{n_1}\cdots b_i^{n_i}\cdots b_j^{n_j}\cdots b_p^{n_p} B_i] \\ &= n_i b_1^{n_1}\cdots b_i^{n_i-1}\cdots b_j^{n_j+1}\cdots b_p^{n_p} + b_1^{n_1}\cdots b_i^{n_i}\cdots b_j^{n_j+1} B_i - \\ &- n_i b_1^{n_1}\cdots b_i^{n_i-1}\cdots b_j^{n_j+1}\cdots b_p + b_1^{n_1}\cdots b_i^{n_i}\cdots b_j^{n_j+1} B_i = 0\end{aligned} \tag{14}$$

If $i = j$ we have

$$\begin{aligned}(\bar{a}_i\bar{b}_i - \bar{b}_i\bar{a}_i) \circ b_1^{n_1}\cdots b_i^{n_i}\cdots b_p^{n_p} &= \bar{a}_i \circ b_1^{n_1}\cdots b_i^{n_i+1}\cdots b_p^{n_p} - \\ &- \bar{b}_i \circ [n_i b_1^{n_1}\cdots b_i^{n_i-1}\cdots b_p^{n_p} + b_1^{n_1}\cdots b_i^{n_i}\cdots b_p^{n_p} B_i] \\ &= (n_i+1) b_1^{n_1}\cdots b_i^{n_i}\cdots b_p^{n_p} + b_1^{n_1}\cdots b_i^{n_i+1}\cdots b_p^{n_p} B_i - n_i b_1^{n_1}\cdots b_i^{n_i}\cdots b_p^{n_p} - \\ &- b_1^{n_1}\cdots b_i^{n_i+1}\cdots b_p^{n_p} B_i = b_1^{n_1}\cdots b_i^{n_i}\cdots b_p^{n_p}\end{aligned}$$

Therefore, $\bar{a}_i\bar{b}_j - \bar{b}_j\bar{a}_i = 1\delta_{ij}$. In an analogous way one proves that $\bar{a}_i\bar{a}_j - \bar{a}_j\bar{a}_i = \bar{b}_i\bar{b}_j - \bar{b}_j\bar{b}_i = 0$. Thus the $G$-representation of $U(H)$, which is also a representation of the quasi-enveloping algebra, is given by the relations

$$\begin{aligned}\bar{a}_i \circ b_1^{n_1}\cdots b_i^{n_i}\cdots b_p^{n_p} &= n_i b_1^{n_1}\cdots b_i^{n_i-1}\cdots b_p^{n_p} + b_1^{n_1}\cdots b_i^{n_i}\cdots b_p^{n_p} B_i \\ \bar{b}_i \circ b_1^{n_1}\cdots b_i^{n_i}\cdots b_p^{n_p} &= b_1^{n_1}\cdots b_i^{n_i+1}\cdots b_p^{n_p}\end{aligned} \tag{15}$$

In the particular case when all $B_i = 0$, we obtain the ladder representation (1) of the creation and annihilation operators.

The representations (15) are realised in the space $V$ of all polynomials of the creation operators. In order to enlarge further the class of representations we proceed to enlarge the representation space. Let $a \in U(H)/J(G)$. Consider the equation

$$a.x = 1 \tag{16}$$

Equation (16) has no solution in $U(H)/J(G)$ for any $a$, i.e. the quasi-enveloping algebra does not contain the inverse of any element different from 1. It has been proved, however, that the quasi-enveloping algebra can be enlarged to a bigger algebra $D$ in such a way that in $D$ every nonzero element has an inverse; that is, $U(H)/J(G)$ can be embedded in a division ring (Gel'fand & Kirrilov, 1965). In $D$ all rational functions of the creation and annihilation operators are well defined. This allows us to enlarge the representation space $V$ to the space $\tilde{V}$ of all rational functions of the creation operators, and in particular to the subspace $P \subset \tilde{V}$ generated by all elements

$$b_1^{n_1} b_2^{n_2}\cdots b_p^{n_p} \tag{17}$$

where $n_i = 0, \pm1, \pm2, \pm3, \ldots, i = ,2,3,\ldots,p$.

The set of all monomials (17) is linearly independent, and therefore spans a basis in $P$. The representation of the Heisenberg algebra generators in $P$ is defined by equation (15) also; however, now $B_i$ is an arbitrary function of $b_i$ from the space $P$ and $n_i$ are arbitrary positive or negative integers.

It is worth pointing out that the ladder representation of the canonical variables in $P$ cannot be extended to a representation of the hole division ring $D$ in the same space. Indeed, suppose this is possible. Let: $a \to \tilde{a} = \theta a$ be a representation of $D$, and let $a^{-1}$ be the inverse of $a_i$. Take the element $b^0 \equiv 1 \in P$. We have

$$(\theta(a_i^{-1})\,\theta(a_i)) \circ b^0 = \theta(a_i^{-1}) \circ (\theta(a_i) \circ b^0) = \theta(a_i^{-1}) \circ 0 = 0$$
$$\theta(a_i^{-1}\,a_i) \circ b^0 = \theta(1).b^0 = b^0 \neq 0$$

Hence, $\theta(a^{-1}\,a_i) \neq \theta(a_i^{-1})\,\theta(a_i)$, and $\theta$ does not preserve the multiplication.

The representations (15) of the canonical variables are infinite dimensional. Apart from some special cases as $B_i = 0$ ($i = 1,\ldots,p$) it is not possible to define a scalar product in $P$ such that $b_i$ will be hermitian conjugate of $a_i$. In spite of this it turns out that the representations (15) are useful for constructing hermitian representations of Lie algebras. This is interesting by itself, since usually the elements of the algebras are expressed as functions of $a_i$ and its hermitian conjugate.

To show that the extension of the representations space $V$ to the space $\tilde{V}$ is relevant, we give an example using the algebra $SU(1,1)$.

*Example*

Consider the algebra $SU(1,1)$. The generators of it, $H_+$, $H_-$ and $H_3$, satisfy the commutation relations [strictly speaking, $H_+$, $H_-$ and $H_3$ are elements from the complex extension of $SU(1,1)$]

$$[H_+,H_3] = -H_+, \qquad [H_-,H_3] = H_-, \qquad [H_+,H_-] = -2H_3 \tag{18}$$

The mapping

$$H_+ \to \tilde{H}_+ = ib_1\,a_2, \qquad H_- \to \tilde{H}_- = ib_2\,a_1$$
$$H_3 \to \tilde{H}_3 = \tfrac{1}{2}(b_1\,a_1 - b_2\,a_2) \tag{19}$$

defines a Lie-homomorphism of $SU(1,1)$ in the algebra $U(H)/J(G)$ generated by two creation and two annihilation operators. Choose the representation space of the canonical variables in equation (15) with $B_1 = B_2 = 0$. Then, the space $V$ of all polynomials of $b_1$ and $b_2$ resolves with respect to the realisation (19) of $SU(1,1)$ into infinite direct sum of finite dimensional irreducible subspaces. Therefore, the space $V$ carries no hermitian representations [$SU(1,1)$ is non-compact]. Consider now the larger space $P$ spanned on all monomials $b_1{}^m b_2{}^n$, with $m$, $n$ being arbitrary positive or negative integers. A simple calculation shows that $P$ resolves also into infinite direct sum of irreducible subspaces $P^j$

$$P = \bigoplus_{j=1}^{\infty} P^j \tag{20}$$

Each space is infinite dimensional and is spanned on the following vectors

$$b_1^{-2j-k} b_2^k \tag{21}$$

where $k = 1, 2, 3, \ldots$

Introduce in every $P^j$ an orthonormal basis

$$e_k^j = \alpha_k^j b_1^{-2j-k} b_2^k \tag{22}$$

and demand the condition

$$\frac{\alpha_k^j}{\alpha_{k+1}^j} = \sqrt{\frac{k+1}{2j+1}} \tag{23}$$

In the metric so introduced, $\overset{+}{H}_3 = H_3$ and $H_+ = \overset{+}{H}_-$. Hence every space $P^j$ carries a hermitian representation of $SU(1,1)$. These representations are well known. In the Bargmann notations this is one of the discrete series $D^{j+}$ (Bargmann, 1949).

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